(for more references see Ref. 1) have made intensive use of the F1 formulation. It was a logical step to extend the already existing two correct formulations to the proposed six in the paper. It must be emphasized that these six formulations, based on the six combinations of vanishing two-end-point variations, are very convenient from the calculation point of view. In addition, by the use of the F4 formulation to develop time-finite elements one obtains "a special form of step-by-step procedure reproduced by the finite elements."[6] A closer look at all of the existing correct formulations built up by vanishing the variations at the end points reveals the fact that, in order to fulfill the 2^n initial conditions of dynamic problems in any direct method, it is desirable to vanish only 2^n variations. From all of the possible formulations only the six suggested in the paper have this characteristic.

It seems to the authors that the argument between us and Professor Smith is purely semantic:

1) Professor Smith prefers the term possibilities over the term used by the authors, formulations. In fact, we use both of them in our paper with somewhat different nuances. However, we agree with Professor Smith that in spite of the many possibilities, Hamilton's Law is only one.

2) It seems that there is some difference in the definition of "physical path" of a system. Professor Smith thinks that the "variations follow the physical path of the system" only for our F1 formulation. Our definition is much broader. It is given in the following:... "we will follow the physical path for defining the variations of the state variables. In other words, the state coordinates and their variations will be built from the same set of admissible functions."[1] It must be emphasized that in the first example,[1] in spite of the different opinion of Professor Smith, the functions of all six formulations of the coordinates are exactly the same. The only difference between them is that in any of them appear the end coordinates which we pretend to know. This fact is especially clear in the F2 formulation, as well as from his final example.

In full agreement with the Comment one can read in the last sentences of the paper: "In other words one can build q and δq from different sets of functions."[1] This possibility was used by the authors[6,7] "to improve the accuracy and the numerical stability of the finite element technique."[1]

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Comment on "Derivation of the Fundamental Equation of Sound Generated by Moving Aerodynamic Surfaces"

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In the Technical Note referred to above, the author attempts to derive the Ffowcs Williams-Hawkings (FW-H) equation by a method which is "simple, direct, physical, and instructive."[1] However, the derivation presented appears to suffer from several deficiencies, which are pointed out below.

1) Both Eqs. (1) and (2) of Ref. 1, which are based on mass and momentum conservation laws, respectively, are written in such a way that they violate these laws. A conventional control volume analysis indicates that the right-hand sides of these equations must be zero. Although many authors have used similar equations, the justification for their use was based on ad hoc, nonrigorous reasoning and the fact that they appeared to give the right answers. It is not obvious at all why one should take particular source terms specified by the author. Indeed, the major achievement of Ffowcs Williams and Hawkings has been to rigorously justify the use of the mass and momentum source terms on the right-hand sides of Eqs. (1) and (2) of Ref. 1.[2] The proper interpretation of these equations is that all of the derivatives on the left-hand sides are generalized derivatives, a fact that the author fails to mention.

2) Having proposed Eqs. (1) and (2), the author then proceeds to derive the exact forms of the source terms on the right side of the so-called generalized mass continuity and momentum conservation laws derived by Ffowcs Williams and Hawkings. The author's argument appears to be artificially constructed to get the right answers. For example, no plausible reasoning is offered to convince the readers that "for the moving surface S, the concentrated mass terms in Eq. (1) may be written as

$$\int_{f<0} \delta(x-\xi)\mathrm{d}m$$

where $\mathrm{d}m$ is a differential mass element at the point $x=\xi$ and integration extends over all the space inside S."[1] Thus a major gap in argument exists in obtaining Eq. (6). Similar reasoning is used to obtain generalized momentum conservation law, Eq. (11). One must again accept that indeed Eq. (8) must be used in place of the point-force term in Eq. (2). The algebraic manipulations from Eq. (9) and leading to Eq. (11) can hardly be termed simple, although they can be simplified considerably.[3]

In short, the author's derivation depends on the acceptance of several results which could be justified rigorously by the kind of analysis proposed by Ffowcs Williams and Hawkings. It is difficult to see that the proposed derivation can improve on the original derivations of the FW-H equation.[2-4]

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Reply by Author to F. Farassat

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1) Dr. Farassat is totally mistaken to think that the right-hand sides of Eqs. (1) and (2) of Ref. 1 "must be zero." The right-hand sides of the mass and momentum conservation equations are zero only when there are no mass discontinuities and no prescribed body forces in the fluid medium. Attention of the discusser is drawn to Eq. (4) of Ref. 2, to Eq. (1.2) of Ref. 3, to the remark following Eq. (2.2.2) of Ref. 4, and to Eq. (6.8) of Ref. 5. A derivation of the continuity equation, Eq. (1) of Ref. 1, is given in the Appendix for the benefit of some readers. In fluid flow problems, it is up to an individual researcher whether to build the mass and force discontinuities into the conservation equations, as is done in the present derivation, or to account for them through boundary conditions as done in the earlier, Ffowcs Williams and Hawkings[6] and Goldstein[3] derivations. It so happens that inclusion of the mass and force distributions into the conservation equations in the present case leads to a simple, direct, physical derivation of the FW-H equation.

2) Writing the concentrated mass terms in Eq. (1), Ref. 1, in case of a moving surface, by the integral

$$\int_{f<0} \delta(x-\xi)\, \mathrm{d}m$$

is allowed by the fact that a surface is a continuum, and the very definition of a definite integral as the limit of a sum. Since almost everyone with some background in mathematics knows about such a basic definition, it hardly needs a justification or a reference. There are no gaps in the argument presented; the logic of the derivation is complete and perfectly valid within the framework of applied mathematics.

The reaction on the part of Dr. Farassat, in face of his personal efforts to obtain a simple derivation of the FW-H equation,[7] is a natural one.

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Appendix

Consider a closed surface S drawn in the region occupied by a moving fluid and fixed in space. Let $\mathrm{d}v$ be an element of the volume contained within S, ρ the density, and q the velocity of this element at any time t. Then, by the law of conservation of mass, the rate of change of mass of the fluid within the control surface S at any time is equal to the mass of the fluid that enters S per unit time plus the mass of the fluid created per unit time inside S due to the motion of the prescribed masses. This gives

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{d}v = -\int_S q\cdot n \mathrm{d}S + \frac{\partial}{\partial t}\int_V \sum_{\xi} Q(\xi)\delta(x-\xi)\mathrm{d}v$$

where $\mathrm{d}S$ is an element of the surface S, n unit outward normal drawn at any point of $\mathrm{d}S$, V the volume of the fluid contained in S, Q the point mass located at he point $x=\xi(t)$ inside S, and $\delta(x)$ the three-dimensional Dirac delta function. Local time derivatives, $\partial/\partial t$, are used because S is fixed in space. Taking the time derivatives inside the integrals and using the divergence theorem gives

$$\int_V \left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho q) - \frac{\partial}{\partial t}\left\{\sum_{\xi} Q(\xi)\delta(x-\xi)\right\}\right]\mathrm{d}v = 0$$

Since V is arbitrary, it yields Eq. (1) of Ref. 1.

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